DIFFERENTIABILITY OF QUERMASSINTEGRALS: A CLASSIFICATION OF CONVEX BODIES

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Abstract. In this paper we characterize the convex bodies in $\mathbb{R}^n$ whose quermassintegrals satisfy certain differentiability properties, which answers a question posed by Bol in 1943 for the 3-dimensional space. This result will have unexpected consequences on the behavior of the roots of the Steiner polynomial: we prove that there exist many convex bodies in $\mathbb{R}^n$, for $n \geq 3$, not satisfying the inradius condition in Teissier's problem on the geometric properties of the roots of the Steiner polynomial.

1. Introduction and main results

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets in the Euclidean space $\mathbb{R}^n$, and let $\mathcal{K}^n_0$ be the subset of $\mathcal{K}^n$ consisting of all convex bodies with non-empty interior. A convex body K is called *strictly convex* if its boundary $\operatorname{bd} K$ does not contain a segment, and *regular* if all its boundary points are regular, i.e., the supporting hyperplane to K at any $x \in \operatorname{bd} K$ is unique. Moreover, let B_n be the n-dimensional unit ball and $\mathbb{S}^{n-1}$ the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$. The volume of a set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{V}(M)$, its closure by $\operatorname{cl} M$, its convex hull by $\operatorname{conv} M$ and its affine hull by $\operatorname{aff} M$.

For two convex bodies $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_0$ and a non-negative real number λ the *outer parallel body of K (relative to E) at distance* λ is the Minkowski sum $K + \lambda E$. For $0 \leq \lambda \leq \mathrm{r}(K;E)$ the *inner parallel body of K (relative to E) at distance* λ is defined as the Minkowski difference

$$K \sim \lambda E = \{x \in \mathbb{R}^n : \lambda E + x \subset K\},$$

where the *relative inradius* $\mathrm{r}(K;E)$ of K with respect to E is given by

$$\mathrm{r}(K;E) = \sup\{r : \exists\, x \in \mathbb{R}^n \text{ with } x + r\,E \subset K\}.$$

When the gauge body $E = B_n$, then $\mathrm{r}(K;B_n)$ is the classical inradius (see [2, p. 59]). Clearly if $\lambda = 0$ the original body K is obtained. Notice that $K \sim \mathrm{r}(K;E)E$ is the set of relative incenters of K, usually called *kernel* of K with respect to E. The dimension of the kernel is always strictly less than n (see [2, p. 59]). Inner parallel bodies and their properties have been studied in [1, 3, 4, 8, 9, 13, 14, 15, 17, 18].

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From now on we will write K_λ to denote the (relative) inner/outer parallel bodies of K, i.e.,

$$K_\lambda := \begin{cases} K \sim |\lambda|\, E & \text{for } -\mathrm{r}(K;E) \le \lambda \le 0,\\ K + \lambda E & \text{for } 0 \le \lambda < \infty. \end{cases} \tag{1.1}$$

Minkowski subtraction is a kind of complementary operation to Minkowski addition: it is easy to prove (see [22, pp. 133–134]) that, in general,

$$(K+E) \sim E = K \quad \text{but} \quad (K \sim E) + E \subseteq K.$$

Equality holds in the above inclusion if and only if E is a *summand* of K, i.e., $K = L + E$ for some $L \in \mathcal{K}^n$. A detailed study of the Minkowski difference and of summands of convex bodies can be found in [22, s. 3.1, 3.2].

The so called *relative Steiner formula* states that the volume of the outer parallel body $K + \lambda E$ is a polynomial of degree n in $\lambda \ge 0$,

$$\mathrm{V}(K+\lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\lambda^i. \tag{1.2}$$

The coefficients $\mathrm{W}_i(K;E)$ are called *relative quermassintegrals* of K, and they are just a special case of the more general *mixed volumes* for which we refer to [22, s. 5.1] and [7, s. 6.2, 6.3]. In particular, we have $\mathrm{W}_0(K;E) = \mathrm{V}(K)$ and $\mathrm{W}_n(K;E) = \mathrm{V}(E)$. If $E = B_n$, then the polynomial in the right hand side of (1.2) becomes the classical *Steiner polynomial*, see [23]. For the sake of brevity we write $f_{K,E}(\lambda) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\lambda^i$ to denote the relative Steiner polynomial of $K \in \mathcal{K}^n$ with respect to $E \in \mathcal{K}^n_0$.

Analogous formulae to (1.2) give the value of the relative i-th quermassintegral of $K + \lambda E$, namely

$$\mathrm{W}_i(K+\lambda E;E) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K;E)\lambda^k, \tag{1.3}$$

for $\lambda \ge 0$ and $i = 0, \ldots, n$ (see [22, (5.1.27) and p. 212]). However, the boundary structure of inner parallel bodies is rather more difficult to control (they are not built just by using a vectorial operation in the Euclidean space), and it also entails that there is no standard way to compute (in general) their volume (quermassintegrals). In [14, 15] the interested reader can find a detailed study of this question, where it is also proved that there is even no chance to give lower/upper bounds for the volume of the inner parallel body of $K \in \mathcal{K}^n$ in terms of the so called *alternating Steiner polynomial* (the polynomial obtained from (1.2) replacing λ by $-\lambda$).

The polynomial expression (1.3) for the quermassintegrals of outer parallel bodies leads to consider the more general function $\mathrm{W}_i(\lambda) := \mathrm{W}_i(K_\lambda;E)$, $\lambda \ge -\mathrm{r}(K;E)$, which is trivially differentiable for $\lambda \ge 0$ (right derivative for $\lambda = 0$). So, the problem arises to determine differentiability properties of $\mathrm{W}_i(\lambda)$ when inner parallel bodies are considered. From the concavity of the family (1.1) and the general Brunn-Minkowski theorem for relative quermassintegrals (see e.g. [22, p. 339]), it is obtained that ${}^{\boldsymbol{\cdot}}\mathrm{W}_i(\lambda) \ge \mathrm{W}_i^{\boldsymbol{\cdot}}(\lambda) \ge (n-i)\mathrm{W}_{i+1}(\lambda)$ for $i = 0, \ldots, n-1$ and for $-\mathrm{r}(K;E) \le \lambda \le 0$. Here ${}^{\boldsymbol{\cdot}}\mathrm{W}_i$ and $\mathrm{W}_i^{\boldsymbol{\cdot}}$ denote, respectively, the left and right derivatives of the function $\mathrm{W}_i(\lambda)$, and for $\lambda = -\mathrm{r}(K;E)$ (respectively, $\lambda = 0$) only the right (left) derivative is considered. In [13] the following natural definition is introduced.

Definition 1.1. Let $E \in \mathcal{K}_0^n$ and let p be an integer, $0 \leq p \leq n-1$. A convex body $K \in \mathcal{K}^n$ *belongs to the class* $\mathcal{R}_p$ if, for all $0 \leq i \leq p$ and for $-\mathrm{r}(K;E) \leq \lambda < \infty$, it holds

$$
{}^{\bullet}\mathrm{W}_i(\lambda) = \mathrm{W}_i^{\bullet}(\lambda) = \mathrm{W}_i'(\lambda) = (n-i)\mathrm{W}_{i+1}(\lambda). \tag{1.4}
$$

Here W_i' denotes the full derivative of W_i since the function is differentiable. Notice that the class $\mathcal{R}_p$ depends on the fixed convex body E. Nevertheless, and for the sake of simplicity, we omit E in the notation. Observe also that (1.4) trivially holds for outer parallel bodies (cf. (1.3)).

It is well-known (see e.g. [1, 17]) that the volume is always differentiable with respect to λ and $\mathrm{V}'(\lambda) = n\mathrm{W}_1(\lambda)$, which implies that $\mathcal{R}_0 = \mathcal{K}^n$. Just from the definition it holds $\mathcal{R}_{i+1} \subset \mathcal{R}_i$, $i = 0, \dots, n-2$, and all these inclusions are strict (see [13]), since there exist $(n-i-1)$-tangential bodies of E lying in $\mathcal{R}_i$ which are not in $\mathcal{R}_{i+1}$ (see Section 2 for the definition). It can be also easily proved that if $K \in \mathcal{R}_p$, then $K + \rho E \in \mathcal{R}_p$ for all $\rho \geq 0$ (see Remark 4.1).

In [8, §23, §29] Hadwiger classified 3-dimensional convex bodies according to the differentiability of quermassintegrals in the above sense, for $E = B_3$, defining three classes, $\mathcal{R}_\alpha$, $\mathcal{R}_\beta$, $\mathcal{R}_\gamma$, which correspond in our notation to $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$, respectively. Earlier, Bol [1, p. 52] had asked for which convex bodies equality in (1.4) is satisfied when $i = 1$, i.e., which are the sets in $\mathcal{R}_\beta$. In [13] the general n-dimensional problem is studied. In particular, it is shown that the smallest class is given by

$$
\mathcal{R}_{n-1} = \left\{ L + \lambda E : L \in \mathcal{K}^n,\ \dim L \leq n-1,\ \lambda \geq 0 \right\} \tag{1.5}
$$

for all $E \in \mathcal{K}_0^n$. Furthermore some necessary conditions for a convex body to belong to the other classes are given.

In this paper we determine the convex bodies lying in the class $\mathcal{R}_{n-2}$:

Theorem 1.1. *Let $E \in \mathcal{K}_0^n$ be regular and strictly convex. A convex body $K \in \mathcal{R}_{n-2}$ if and only if K is (an outer parallel body of) a cap-body of any set lying in $\mathcal{R}_{n-1}$, satisfying the condition*

$$
\mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*) \tag{1.6}
$$

for $-\mathrm{r}(K;E) \leq \lambda \leq 0$.

Here $\mathcal{U}_0(K)$ denotes the set of the so called 0-extreme vectors of K, and K^* is its relative form body (see Section 2 for definitions). A cap-body of E is the convex hull of E and countably many points such that the line segment joining any pair of those points intersects E.

Although condition (1.6) is technical and possibly not easy to interpret, there is a lot of geometry hidden behind it: Corollary 4.1 and Remark 4.2 collect geometrical properties implied by this condition, whereas Remark 2.1 shows a convex body not satisfying (1.6). Notice also that the class $\mathcal{R}_{n-1}$ can be characterized in full generality for an arbitrary $E \in \mathcal{K}_0^n$, since the proof relies ultimately on formulae (1.3). However, the technical requirements for the proof of Theorem 1.1 do not allow a direct extension to an arbitrary $E \in \mathcal{K}_0^n$, and the regularity and strict convexity are needed.

Theorem 1.1 answers the question posed by Bol in [1], and together with characterization (1.5) of $\mathcal{R}_{n-1}$ we set the complete description of the bodies lying in the classes defined by Hadwiger. To describe the solution we keep the original notation in [8] for the classes $\mathcal{R}_p$ in dimension $n = 3$ with $E = B_3$, namely, $\mathcal{R}_\beta = \mathcal{R}_1$ and $\mathcal{R}_\gamma = \mathcal{R}_2$.

Corollary 1.1. *The only sets in $\mathcal{R}_\gamma$ are outer parallel bodies of k-dimensional convex bodies, for $0 \le k \le 2$, i.e.,*

$$\mathcal{R}_\gamma = \bigl\{L + \lambda B_3 : L \in \mathcal{K}^3,\ \dim L \le 2,\ \lambda \ge 0\bigr\}.$$

A convex body $K \in \mathcal{R}_\beta$ if and only if K is (an outer parallel body of) a cap-body of any set lying in $\mathcal{R}_\gamma$, satisfying (1.6) *for* $-\mathrm{r}(K;B_3) \le \lambda \le 0$.

Theorem 1.1 has a consequence on the behavior of the roots of the Steiner polynomial $f_{K,E}(\lambda)$. Based on a problem posed by Teissier in [24], Sangwine-Yager stated in [19] the following conjecture (see also [20, p. 65]).

Conjecture 1.1. *Let $K, E \in \mathcal{K}^n$. If $a_1 \le \cdots \le a_n$ are the real parts of the roots of $f_{K,E}(\lambda)$, then $a_1 \le -\mathrm{R}(K;E) \le -\mathrm{r}(K;E) \le a_n \le 0$.*

Here $\mathrm{R}(K;E) = \min\{R : \exists\, x \in \mathbb{R}^n \text{ with } K \subseteq x + R\,E\}$ is the *relative circumradius* of K with respect to E. The full conjecture is known to be true in dimension $n = 2$, but false in general dimension. Precisely, in [10, 11] it is shown that:

i) $a_i \le 0$ for all $i = 1, \dots, n$ when $n \le 9$, whereas for $n = 12$ there are sets K for which $f_{K,B_{12}}(\lambda)$ has roots with strictly positive real part.
ii) for $n = 3$ and $E = B_3$ there exist convex bodies K such that $a_i > \mathrm{R}(K;B_3)$ for all $i = 1, \dots, n$.
iii) for $n = 3$ there are convex bodies K, E such that $a_i < \mathrm{r}(K;E)$ for all $i = 1, \dots, n$.

However, for the counterexamples of case iii) the gauge body is not the ball. So the conjecture might be true (for some value of the dimension) when the gauge body is $E = B_n$. As a consequence of Theorem 1.1 we are able to prove that it is not so for $n = 3$ and $E = B_3$.

Theorem 1.2. *There exists $K \in \mathcal{K}^3_0$ such that all the roots of $f_{K,B_3}(\lambda)$ are real and strictly less than $-\mathrm{r}(K;B_3)$.*

The paper is organized as follows. In Section 2 we give the needed notation and definitions, as well as preliminary lemmas. The proof of Theorem 1.1 relies on results which state properties of the sets lying in $\mathcal{R}_{n-2}$; they are contained in Section 3. The last two sections are mainly devoted to the proofs of Theorem 1.1 and Theorem 1.2, respectively.

2. Additional notation and preliminary lemmas

For convex bodies $K_1, \dots, K_m \in \mathcal{K}^n$ and real numbers $\lambda_1, \dots, \lambda_m \ge 0$, the volume of the linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is expressed as a polynomial of degree n in the variables $\lambda_1, \dots, \lambda_m$,

$$\mathrm{V}\bigl(\lambda_1 K_1 + \cdots + \lambda_m K_m\bigr) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \mathrm{V}(K_{i_1}, \dots, K_{i_n})\lambda_{i_1} \cdots \lambda_{i_n},$$

whose coefficients $\mathrm{V}(K_{i_1}, \dots, K_{i_n})$ are the *mixed volumes* of $K_1, \dots, K_m$. This formula (and hence mixed volumes) extends the relative Steiner formula (1.2) (relative quermassintegrals). In particular, the i-th quermassintegral

$$\mathrm{W}_i(K;E) = \mathrm{V}\bigl(K, \overset{(n-i)}{\ldots}, K, E, \overset{(i)}{\ldots}, E\bigr). \tag{2.1}$$

The *mixed surface area measure* $\mathrm{S}(K_1,\dots,K_{n-1};\cdot)$, for $K_1,\dots,K_{n-1}\in\mathcal{K}^n$, is the unique finite Borel measure on $\mathbb{S}^{n-1}$ such that for all $K\in\mathcal{K}^n$,

$$\mathrm{V}(K,K_1,\dots,K_{n-1})=\frac{1}{n}\int_{\mathbb{S}^{n-1}}h(K,u)\,d\mathrm{S}(K_1,\dots,K_{n-1};u). \tag{2.2}$$

Here $h(K,u)=\sup\bigl\{\langle x,u\rangle : x\in K\bigr\}$, $u\in\mathbb{R}^n$, denotes the *support function* of the set $K\in\mathcal{K}^n$ (see e.g. [22, s. 1.7]). For the sake of brevity we will write $\bigl(K_1[r_1],\dots,K_m[r_m]\bigr)\equiv\bigl(K_1,\overset{(r_1)}{\dots},K_1,\dots,K_m,\overset{(r_m)}{\dots},K_m\bigr)$. For a deep study of mixed volumes and mixed surface area measures we refer to [22, s. 5.1].

A vector $u\in\mathbb{S}^{n-1}$ is an *r-extreme normal vector* of K, $0\le r\le n-1$, if it cannot be written as $u=u_1+\cdots+u_{r+2}$, with u_i linearly independent normal vectors at one and the same boundary point of K. We write $\mathcal{U}_r(K)$ to denote the set of r-extreme normal vectors of K and it is clear that $\mathcal{U}_r(K)\subseteq\mathcal{U}_s(K)$ for any $0\le r<s\le n-1$. Polytopes provide an easy example in order to distinguish the different types of extreme vectors: the normal vectors at any point of the relative interior of an $(n-i)$-face is an $(i-1)$-extreme vector. Notice also that if $x\in\mathrm{bd}\,K$ is a regular point, then the only outer normal vector at x is 0-extreme; indeed, 0-extreme vectors are normal vectors either at regular points or at limits of regular points. We mention also that, in general, the set $\mathcal{U}_0(K)$ is not closed, as shown in Figure 1, the intersection of a cylinder and two suitable subspaces: all normal vectors at any point in the circle C are 0-extreme except for u_0.

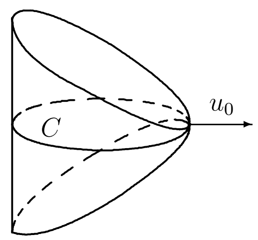

FIGURE 1. $\mathcal{U}_0(K)$ is not closed (figure is taken from [18, p.28]).

A support plane is said to be r-extreme if its outer normal vector is r-extreme. For a detailed study of r-extreme vectors we refer to [22, s. 2.2] (see also [18, ch. 2]).

The (relative) *form body* of a convex body $K\in\mathcal{K}^n_0$ *with respect to* $E\in\mathcal{K}^n_0$, denoted by K^*, is defined as (see e.g. [4])

$$K^*=\bigcap_{u\in\mathcal{U}_0(K)}\bigl\{x:\langle x,u\rangle\le h(E,u)\bigr\}.$$

Notice that $(K^*)^*=K^*$. For an alternative (equivalent) definition of form body see [22, p. 321]. Notice that K^* depends also on the gauge body E. Nevertheless, and for the sake of simplicity, we again omit E in the notation.

Finally, a convex body $K\in\mathcal{K}^n$ containing $E\in\mathcal{K}^n_0$ is called a *p-tangential body* of E, $p\in\{0,\dots,n-1\}$, if each $(n-p-1)$-extreme support plane of K supports E. For further characterizations and properties of p-tangential bodies we refer to [22, Section 2.2]. It is easy to see that a 0-tangential body of E is E itself and each p-tangential body is also a q-tangential body for $p<q\le n-1$. Moreover, 1-tangential bodies are just the cap-bodies (see [22, p. 76]). We will briefly call *tangential body*

an $(n-1)$-tangential body. Notice that the form body K^* of a convex body K is always a tangential body of E, and there exist p-tangential bodies of E which are not $(p-1)$-tangential bodies of E (see e.g. [6, p. 163] or [10, Proof Th. 1.2]).

2.1. **Some preliminary results.** In [13, Lemma 2.1] it is shown that if $E\in\mathcal{K}^n_0$ is a regular convex body then, for any $K\in\mathcal{K}^n$,

$$\mathrm{cl}\,\mathcal{U}_0(K)=\mathcal{U}_0(K^*). \tag{2.3}$$

Moreover, it is known that for any $K\in\mathcal{K}^n$ and $-\mathrm{r}<\lambda\leq 0$ it holds (see [18, Lemma 4.5])

$$\mathcal{U}_0(K_\lambda)\subseteq\mathcal{U}_0(K), \tag{2.4}$$

and for any $K,L\in\mathcal{K}^n$ we have (see [18, Lemma 2.4])

$$\mathcal{U}_0(K)\cup\mathcal{U}_0(L)\subseteq\mathcal{U}_0(K+L) \tag{2.5}$$

and (see [15, Lemma 3.1])

$$\mathcal{U}_0(K+L)=\mathcal{U}_0(K+\lambda L),\quad \lambda>0. \tag{2.6}$$

We show a similar property regarding the $(n-2)$-extreme vectors of the sum.

Lemma 2.1. *Let $K,L\in\mathcal{K}^n$. Then for any $\mu>0$ it holds*

$$\mathrm{cl}\,\mathcal{U}_{n-2}(K+\mu L)=\mathrm{cl}\,\mathcal{U}_{n-2}(K)\cup\mathrm{cl}\,\mathcal{U}_{n-2}(L).$$

Proof. Let $E\in\mathcal{K}^n$ be a regular and strictly convex body. Then it is known (see [21, pp. 135–136]) that for any $K\in\mathcal{K}^n$ and for $i=0,\dots,n-1$,

$$\mathrm{supp}\,\mathrm{S}\big(K[n-i-1],E[i];\cdot\big)=\mathrm{cl}\,\mathcal{U}_i(K). \tag{2.7}$$

Here $\mathrm{supp}\,\nu$ denotes the support of the measure ν.

In particular, $\mathrm{cl}\,\mathcal{U}_{n-2}(K+\mu L)=\mathrm{supp}\,\mathrm{S}\big(K+\mu L,E[n-2];\cdot\big)$, and the linearity of the surface area measure in each argument (see [22, p. 279]), i.e.,

$$\mathrm{S}\big(K+\mu L,E[n-2];\cdot\big)=\mathrm{S}\big(K,E[n-2];\cdot\big)+\mu\mathrm{S}\big(L,E[n-2];\cdot\big),$$

allows us to conclude that

$$\begin{aligned}\mathrm{supp}\,\mathrm{S}\big(K+\mu L,E[n-2];\cdot\big)&=\mathrm{supp}\,\mathrm{S}\big(K,E[n-2];\cdot\big)\cup\mathrm{supp}\,\mathrm{S}\big(L,E[n-2];\cdot\big)\\&=\mathrm{cl}\,\mathcal{U}_{n-2}(K)\cup\mathrm{cl}\,\mathcal{U}_{n-2}(L),\end{aligned}$$

as required. □

From now on we will write $\mathrm{r}=\mathrm{r}(K;E)$ for the sake of brevity, unless it is not clear from the context.

It is known (see [18, Lemma 4.8]) that it always holds

$$K\supseteq K_\lambda+|\lambda|\,K^* \tag{2.8}$$

for any $K\in\mathcal{K}^n$, $E\in\mathcal{K}^n_0$ and all $-\mathrm{r}<\lambda\leq 0$. Taking into account that $\lim_{\lambda\to-\mathrm{r}}K_\lambda=K_{-\mathrm{r}}$ (see e.g. [9, s. 4.3.1]) and the continuity of the support function on $\mathcal{K}^n$ with respect to the Hausdorff metric, it is easy to see that whenever a relation of the type (2.8) holds for $-\mathrm{r}<\lambda\leq 0$, then also the case $\lambda=-\mathrm{r}$ is covered:

Lemma 2.2. *Let $K\in\mathcal{K}^n$ and let $E\in\mathcal{K}^n_0$. If $K=K_\lambda+|\lambda|\,K^*$ for every $-\mathrm{r}<\lambda\leq 0$, then it also holds $K=K_{-\mathrm{r}}+\mathrm{r}K^*$.*

In [15, Theorem 2.2] the convex bodies $K\in\mathcal{K}^n$ satisfying $K=K_\lambda+|\lambda|\,K^*$ for every $-\mathrm{r}\leq\lambda\leq 0$ are characterized:

Theorem 2.1 ([15, Theorem 2.2])**.** *Let $K, E \in \mathcal{K}_0^n$ with E regular. Then $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \le \lambda \le 0$ if and only if K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ satisfying that for all $-\mathrm{r} \le \lambda \le 0$,*

$$\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*). \tag{2.9}$$

It can be proved that this characterization may be expressed in terms of condition (1.6) in the following way.

Theorem 2.2. *Let $K, E \in \mathcal{K}_0^n$ with E regular. Then $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \le \lambda \le 0$ if and only if K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ satisfying condition* (1.6) *for all $-\mathrm{r} \le \lambda \le 0$.*

We would like to notice that conditions (1.6) and (2.9) are equivalent in this particular case, because the special fact that $K = K_\lambda + |\lambda| K^*$ implies that $\mathcal{U}_0(K)$ is closed (cf. also Figure 1).

In order to show Theorem 2.2 we need the following result, proved in [18]. We write $K_\lambda^* = (K_\lambda)^*$ to denote the form body of the inner parallel body of K at distance $|\lambda|$, $-\mathrm{r} < \lambda \le 0$; notice that $K_{-\mathrm{r}}^*$ can be unbounded or empty.

Lemma 2.3 ([18, Corollary to Lemma 4.8 and Lemma 4.9])**.** *Let $K \in \mathcal{K}^n$ and let $E \in \mathcal{K}_0^n$. If for every $-\mathrm{r} < \lambda \le 0$*

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K_\lambda + K_\lambda^*), \tag{2.10}$$

then the following properties hold:

i) *$K \subseteq K_\lambda + |\lambda|\, K_\lambda^*$ and*

ii) *$\dfrac{d}{d\lambda} h(K_\lambda, u) = h(K_\lambda^*, u)$ for every $u \in \mathbb{S}^{n-1}$.*

We point out that, for all $u \in \mathbb{S}^{n-1}$, since $h(K_\lambda, u)$ is concave and monotonous with respect to $\lambda \in (-\mathrm{r}, 0]$, the derivative of $h(K_\lambda, u)$ always exists almost everywhere for $\lambda \le 0$ (cf. [18, Lemma 4.9]).

Notice also that condition (2.10) differs from condition (1.6) (when $\lambda \neq 0$), and none of these implies the other. A detailed study about condition (2.10) can be found in [18, p. 39]

Proof of Theorem 2.2. If $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \le \lambda \le 0$, then Theorem 2.1 yields that K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ satisfying condition (2.9) which, together with (2.5), (2.3) and (2.4) gives

$$\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*) \supseteq \mathcal{U}_0(K_\lambda) \cup \mathcal{U}_0(K^*) = \mathcal{U}_0(K_\lambda) \cup \mathrm{cl}\,\mathcal{U}_0(K) = \mathrm{cl}\,\mathcal{U}_0(K).$$

This implies that $\mathcal{U}_0(K) = \mathrm{cl}\,\mathcal{U}_0(K)$, and hence we get condition (1.6).

It remains to show that if K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ satisfying (1.6) for all $-\mathrm{r} \le \lambda \le 0$, then $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \le \lambda \le 0$. Since K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$, it is known (see [15, Lemma 3.2]) that

$$\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K) \quad \text{for } -\mathrm{r} < \lambda \le 0, \tag{2.11}$$

which implies that $K^* = K_\lambda^*$ for $-\mathrm{r} < \lambda \le 0$. Then

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*) = \mathcal{U}_0(K_\lambda + K_\lambda^*)$$

for $-\mathrm{r} < \lambda \le 0$, and we can apply Lemma 2.3 ii) to get

$$\frac{d}{d\lambda} h(K_\lambda, u) = h(K_\lambda^*, u) = h(K^*, u), \quad \text{for } -\mathrm{r} < \lambda \le 0,\ u \in \mathbb{S}^{n-1}.$$

Now for each $u \in \mathbb{S}^{n-1}$, the function $f(\lambda) = h(K,u) - h(K_\lambda,u) + \lambda h(K^*,u)$ is absolutely continuous (since $h(K_\lambda,u)$ is concave in λ, see [7, Theorem 1.1]), almost everywhere differentiable and satisfies $f'(\lambda) = 0$ for $-\mathrm{r} < \lambda \le 0$ and $f(0) = 0$. Then $f \equiv 0$ in $-\mathrm{r} < \lambda \le 0$, i.e.,

$$h(K,u) = h(K_\lambda,u) + |\lambda|\, h(K^*,u) = h(K_\lambda + |\lambda|\, K^*,u)$$

which implies that $K = K_\lambda + |\lambda|\, K^*$ for $-\mathrm{r} < \lambda \le 0$. Lemma 2.2 ensures that $K = K_\lambda + |\lambda|\, K^*$ in the full range $-\mathrm{r} \le \lambda \le 0$. □

Next lemma shows, roughly speaking, that the property of being a cap-body is "transfered" to the inner parallel bodies and the form body.

Lemma 2.4. *Let $E \in \mathcal{K}^n_0$ be a regular convex body and let $K \in \mathcal{K}^n$ be a cap-body of $K_{-\mathrm{r}} + \mathrm{r}E$ satisfying condition* (1.6) *for all $-\mathrm{r} \le \lambda \le 0$. Then*

i) *K^* is a cap-body of E and*
ii) *K_λ is a cap-body of $K_{-\mathrm{r}} + (\mathrm{r}+\lambda)E$.*

Proof. i) Since K is a cap-body of $K_{-\mathrm{r}} + \mathrm{r}E$ satisfying (1.6), Theorem 2.2 ensures that $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \le \lambda \le 0$. Then by Lemma 2.1 we get, in particular, that

$$\mathrm{cl}\,\mathcal{U}_{n-2}(K^*) \subseteq \mathrm{cl}\,\mathcal{U}_{n-2}\big(K_\lambda + |\lambda|\, K^*\big) = \mathrm{cl}\,\mathcal{U}_{n-2}(K).$$

From the regularity of E we know that $\mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K^*)$ (cf. (2.3)) and moreover, $K_{-\mathrm{r}} + \mathrm{r}E$ is also regular. Hence, since K is a cap-body of $K_{-\mathrm{r}} + \mathrm{r}E$, it follows that $\mathcal{U}_0(K) = \mathcal{U}_{n-2}(K)$ and we get

$$\mathcal{U}_{n-2}(K^*) \subseteq \mathrm{cl}\,\mathcal{U}_{n-2}(K^*) \subseteq \mathrm{cl}\,\mathcal{U}_{n-2}(K) = \mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K^*).$$

Then K^* is a tangential body of E satisfying that $\mathcal{U}_0(K^*) = \mathcal{U}_{n-2}(K^*)$, which shows that K^* is a cap-body of E.

Now we prove ii). In [15, (3.8)] it is shown that whenever we have the decomposition $K = K_\lambda + |\lambda|\, K^*$ for $K \in \mathcal{K}^n$ and all $\lambda \in [-\mathrm{r}, 0]$, then

$$K_\lambda = K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^* \tag{2.12}$$

for all $\lambda \in [-\mathrm{r}, 0]$. By Theorem 2.2, $K = K_\lambda + |\lambda|\, K^*$ for every $\lambda \in [-\mathrm{r}, 0]$, and thus (2.12) holds. Moreover, since K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$, we know that $\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K)$, i.e., $K^* = K^*_\lambda$, for $-\mathrm{r} < \lambda \le 0$ (cf. (2.11)). Thus, we get $K_\lambda = K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^* = K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^*_\lambda$, and Theorem 2.2 applied to K_λ ensures that all inner parallel bodies of K are tangential bodies of $K_{-\mathrm{r}} + (\mathrm{r}+\lambda)E$. It remains to show that moreover, it is a *cap-body* of $K_{-\mathrm{r}} + (\mathrm{r}+\lambda)E$. From (2.12) it follows that for any $-\mathrm{r} \le \lambda \le 0$ and every $u \in \mathcal{U}_0(K)$

$$h(K_\lambda,u) = h(K_{-\mathrm{r}},u) + (\mathrm{r}+\lambda)h(K^*,u) = h(K_{-\mathrm{r}},u) + (\mathrm{r}+\lambda)h(E,u).$$

Then it is enough to prove that

$$\mathcal{U}_0(K_\lambda) = \mathcal{U}_{n-2}(K_\lambda) \quad \text{for every } -\mathrm{r} < \lambda \le 0, \tag{2.13}$$

since these last two assertions, together with (2.11), imply that

$$\begin{gathered} h(K_\lambda,u) = h(K_{-\mathrm{r}},u) + (\mathrm{r}+\lambda)h(E,u) = h\big(K_{-\mathrm{r}} + (\mathrm{r}+\lambda)E,u\big) \\ \text{for every } \ u \in \mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda) = \mathcal{U}_{n-2}(K_\lambda) \end{gathered}$$

and any $-\mathrm{r} < \lambda \le 0$; it shows that K_λ is a cap-body of $K_{-\mathrm{r}} + (\mathrm{r}+\lambda)E$.

We prove (2.13). Using Lemma 2.1 twice and (2.12) we get

$$\begin{aligned}\mathrm{cl}\,\mathcal{U}_{n-2}(K) &= \mathrm{cl}\,\mathcal{U}_{n-2}(K_{-\mathrm{r}} + \mathrm{r}K^*) = \mathrm{cl}\,\mathcal{U}_{n-2}(K_{-\mathrm{r}}) \cup \mathrm{cl}\,\mathcal{U}_{n-2}(K^*) \\ &= \mathrm{cl}\,\mathcal{U}_{n-2}\big(K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^*\big) = \mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda).\end{aligned}$$

Moreover, since K is a cap-body of a regular convex body, it holds that $\mathcal{U}_0(K) = \mathcal{U}_{n-2}(K)$, and with (2.11) we can conclude that

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_0(K) = \mathrm{cl}\,\mathcal{U}_{n-2}(K) = \mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda)$$

for every $-\mathrm{r} < \lambda \leq 0$. Finally we show that the closures can be omitted. Indeed, since $K_\lambda = K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^*$, then by (2.5) we get that, in particular, $\mathcal{U}_0(K^*) \subseteq \mathcal{U}_0(K_\lambda)$. Thus, together with (2.11) and (2.3) we obtain that

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_0(K) = \mathcal{U}_0(K^*) \subseteq \mathcal{U}_0(K_\lambda) \subseteq \mathcal{U}_{n-2}(K_\lambda) \subseteq \mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda).$$

Since $\mathrm{cl}\,\mathcal{U}_0(K_\lambda) = \mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda)$, the inclusions in the middle also coincide, i.e., $\mathcal{U}_0(K_\lambda) = \mathcal{U}_{n-2}(K_\lambda)$, as required. □

Remark 2.1. Note that condition (1.6) cannot be omitted in the above lemma, as the following example, provided in [15, Remark 3.2], shows. Let $\sigma \subset \mathbb{R}^3$ be a line segment of length not smaller than 2 and take a point x lying outside the solid cylinder with circular cross section of radius 1 and axis the line $\mathrm{aff}\,\sigma$. The convex body $K = \mathrm{conv}\{\sigma + B_3, x\}$ (see Figure 2, left) is a cap-body of $\sigma + B_3 = K_{-1} + B_3$, since $\mathrm{r}(K; B_3) = 1$ and $K_{-1} = \sigma$. In addition, K^* is the convex hull of B_3 and a suitable segment (see Figure 2, right), which yields that $K_{-1} + K^*$ has more 0-extreme vectors than K, namely, $(0,0,1) \in \mathcal{U}_0(K_{-1} + K^*) \backslash \mathcal{U}_0(K)$. Thus condition (1.6) does not hold. Besides, item i) in Lemma 2.4 is not fulfilled.

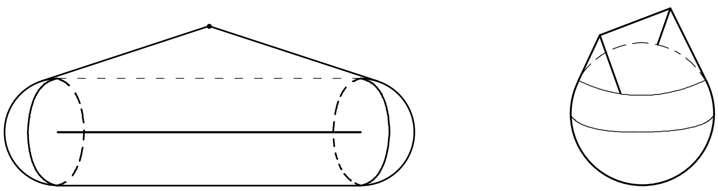

FIGURE 2. A cap-body of $K_{-\mathrm{r}} + \mathrm{r}B_3$ not satisfying (1.6) and its form body.

The following lemma shows that condition $K = K_\lambda + |\lambda|\, K^*$ can be related to the linearity of the family K_λ. Again condition (1.6) plays a crucial role.

Lemma 2.5. *Let $E \in \mathcal{K}_0^n$ be a regular convex body and $K \in \mathcal{K}^n$. Then $K = K_\lambda + |\lambda|\, K^*$ for any $-\mathrm{r} \leq \lambda \leq 0$ if and only if*

$$K_\lambda = \frac{|\lambda|}{\mathrm{r}} K_{-\mathrm{r}} + \left(1 - \frac{|\lambda|}{\mathrm{r}}\right) K \tag{2.14}$$

for every $-\mathrm{r} \leq \lambda \leq 0$ and condition (1.6) *holds.*

Proof. First we assume that $K = K_\lambda + |\lambda|\, K^*$. Then from Theorem 2.2 we get condition (1.6), and it also holds that $K_\lambda = K_{-\mathrm{r}} + \big(\mathrm{r} - |\lambda|\big)K^*$ for any $-\mathrm{r} \leq \lambda \leq 0$.

Thus,

$$\begin{aligned}\frac{|\lambda|}{\mathrm{r}}K_{-\mathrm{r}}+\left(1-\frac{|\lambda|}{\mathrm{r}}\right)K&=\frac{|\lambda|}{\mathrm{r}}K_{-\mathrm{r}}+\left(1-\frac{|\lambda|}{\mathrm{r}}\right)\bigl(K_\lambda+|\lambda|\,K^*\bigr)\\&=\frac{|\lambda|}{\mathrm{r}}K_{-\mathrm{r}}+\frac{\mathrm{r}-|\lambda|}{\mathrm{r}}\,K_\lambda+\frac{|\lambda|}{\mathrm{r}}\bigl(\mathrm{r}-|\lambda|\bigr)K^*\\&=\frac{|\lambda|}{\mathrm{r}}\Bigl[K_{-\mathrm{r}}+\bigl(\mathrm{r}-|\lambda|\bigr)K^*\Bigr]+\frac{\mathrm{r}-|\lambda|}{\mathrm{r}}K_\lambda\\&=\frac{|\lambda|}{\mathrm{r}}K_\lambda+\frac{\mathrm{r}-|\lambda|}{\mathrm{r}}K_\lambda=K_\lambda.\end{aligned}$$

Conversely, now we assume (2.14) and (1.6). Using (2.4), (2.5) and (2.6) we get from (2.14) that

$$\mathcal{U}_0(K_\lambda)\supseteq\mathcal{U}_0(K_{-\mathrm{r}})\cup\mathcal{U}_0(K)\supseteq\mathcal{U}_0(K)\supseteq\mathcal{U}_0(K_\lambda)$$

for all $-\mathrm{r}<\lambda\leq 0$, i.e., $\mathcal{U}_0(K_\lambda)=\mathcal{U}_0(K)$. On the other hand, it is known (see [18, Lemma 4.4]) that if $u\in\mathcal{U}_0(K_\lambda)$ then $h(K_\lambda,u)=h(K,u)-|\lambda|\,h(E,u)$. Thus for every $u\in\mathcal{U}_0(K)=\mathcal{U}_0(K_\lambda)$ we have

$$h(K,u)-|\lambda|\,h(E,u)=h(K_\lambda,u)=\frac{|\lambda|}{\mathrm{r}}h(K_{-\mathrm{r}},u)+\left(1-\frac{|\lambda|}{\mathrm{r}}\right)h(K,u)$$

for any $-\mathrm{r}<\lambda\leq 0$, i.e.,

$$\frac{|\lambda|}{\mathrm{r}}h(K,u)=\frac{|\lambda|}{\mathrm{r}}h(K_{-\mathrm{r}},u)+|\lambda|\,h(E,u),$$

which leads to

$$h(K,u)=h(K_{-\mathrm{r}},u)+\mathrm{r}\,h(E,u)=h(K_{-\mathrm{r}}+\mathrm{r}E,u)$$

for every $u\in\mathcal{U}_0(K)$. It shows that K is a tangential body of $K_{-\mathrm{r}}+\mathrm{r}E$ satisfying, by hypothesis, (1.6). Theorem 2.2 allows us to conclude that $K=K_\lambda+|\lambda|\,K^*$. □

3. Towards the geometry of $\mathcal{R}_{n-2}$

Our aim in this section is to show that condition (1.6) is necessary for a convex body to lie in $\mathcal{R}_{n-2}$, as well as characterize the tangential bodies of $K_{-\mathrm{r}}+\mathrm{r}E$ lying in $\mathcal{R}_{n-2}$. We start proving the necessity of (1.6).

Proposition 3.1. *Let $E\in\mathcal{K}^n_0$ be a regular and strictly convex body and let $K\in\mathcal{R}_{n-2}$. Then $\mathrm{cl}\,\mathcal{U}_0(K)=\mathcal{U}_0(K_\lambda+K^*)$ for every $-\mathrm{r}<\lambda\leq 0$.*

Proof. First notice that since E is regular then (cf. (2.3) and (2.5))

$$\mathrm{cl}\,\mathcal{U}_0(K)=\mathcal{U}_0(K^*)\subseteq\mathcal{U}_0(K_\lambda)\cup\mathcal{U}_0(K^*)\subseteq\mathcal{U}_0(K_\lambda+K^*),$$

and we have to show the reverse inclusion. Any $K\in\mathcal{R}_{n-1}$ is an outer parallel body of a lower dimensional set (cf. (1.5)), and hence condition (1.6) holds trivially since K is regular, i.e., since $\mathcal{U}_0(K)=\mathbb{S}^{n-1}$. Thus we assume that $K\in\mathcal{R}_{n-2}\backslash\mathcal{R}_{n-1}$ and it follows from [13, Theorem 1.2, iv)] that

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda)=\mathrm{cl}\,\mathcal{U}_1(K_\lambda)=\cdots=\mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda)\tag{3.1}$$

for every $-\mathrm{r}<\lambda\leq 0$. Now, since E is regular and strictly convex, it holds $\mathrm{cl}\,\mathcal{U}_0(K_\lambda+K^*)=\mathrm{supp}\,\mathrm{S}\bigl(K_\lambda+K^*[n-1];\cdot\bigr)$ (cf. (2.7)). The linearity of the surface area measure in each argument yields

$$\operatorname{supp} \mathrm{S}\big(K_\lambda + K^*[n-1]; \cdot\big) = \operatorname{supp} \mathrm{S}\big(K_\lambda[n-1]; \cdot\big) \cup \operatorname{supp} \mathrm{S}\big(K^*[n-1]; \cdot\big) \\ \cup \left[\bigcup_{i=1}^{n-2} \operatorname{supp} \mathrm{S}\big(K_\lambda[i], K^*[n-i-1]; \cdot\big)\right],$$

and hence we get that

$$\mathcal{U}_0(K_\lambda + K^*) \subseteq \operatorname{cl}\mathcal{U}_0(K_\lambda + K^*) = \operatorname{supp} \mathrm{S}\big(K_\lambda + K^*[n-1]; \cdot\big) \\ = \operatorname{cl}\mathcal{U}_0(K_\lambda) \cup \operatorname{cl}\mathcal{U}_0(K^*) \cup \left[\bigcup_{i=1}^{n-2} \operatorname{supp} \mathrm{S}\big(K_\lambda[i], K^*[n-i-1]; \cdot\big)\right].$$

In [21, Lemma 3.4] it is shown that, for any $n-1$ sets $K, K_1, \dots, K_{n-2} \in \mathcal{K}^n$,

$$\operatorname{supp} \mathrm{S}\big(K, K_1, \dots, K_{n-2}; \cdot\big) \subseteq \operatorname{supp} \mathrm{S}\big(E, K_1, \dots, K_{n-2}; \cdot\big),$$

provided $E \in \mathcal{K}_0^n$ is regular and strictly convex. Hence, we have

$$\operatorname{supp} \mathrm{S}\big(K_\lambda[i], K^*[n-i-1]; \cdot\big) \subseteq \operatorname{supp} \mathrm{S}\big(K_\lambda[i], E[n-i-1]; \cdot\big) = \operatorname{cl}\mathcal{U}_{n-i-1}(K_\lambda)$$

for $i = 1, \dots, n-2$ (cf. (2.7)) and thus, together with (3.1), (2.3) and (2.4) it follows that

$$\mathcal{U}_0(K_\lambda + K^*) \subseteq \operatorname{cl}\mathcal{U}_0(K_\lambda) \cup \operatorname{cl}\mathcal{U}_0(K^*) \cup \left[\bigcup_{i=1}^{n-2} \operatorname{cl}\mathcal{U}_{n-i-1}(K_\lambda)\right] \\ = \operatorname{cl}\mathcal{U}_0(K_\lambda) \cup \operatorname{cl}\mathcal{U}_0(K^*) = \operatorname{cl}\mathcal{U}_0(K_\lambda) \cup \operatorname{cl}\mathcal{U}_0(K) = \operatorname{cl}\mathcal{U}_0(K),$$

which shows the result. □

Proposition 3.2. *Let $E \in \mathcal{K}_0^n$ be a regular and strictly convex body and let $K \in \mathcal{K}^n$ be a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$. Then, $K \in \mathcal{R}_{n-2}$ if and only if K is a cap-body of $K_{-\mathrm{r}} + \mathrm{r}E$ satisfying* (1.6) *for $-\mathrm{r} \le \lambda \le 0$.*

Proof. First we assume that $K \in \mathcal{R}_{n-2}$. Then Proposition 3.1 ensures that K satisfies (1.6) for $-\mathrm{r} < \lambda \le 0$. On the other hand, since K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$, we have $\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K)$ for all $-\mathrm{r} < \lambda \le 0$ (cf. (2.11)) and hence $K^* = K_\lambda^*$ for $-\mathrm{r} < \lambda \le 0$. Thus condition (1.6) can be rewritten as $\operatorname{cl}\mathcal{U}_0(K_\lambda) = \mathcal{U}_0(K_\lambda + K_\lambda^*)$, and by Lemma 2.3 i) we have $K \subseteq K_\lambda + |\lambda|\, K_\lambda^*$ for $-\mathrm{r} < \lambda \le 0$. Since $K \supseteq K_\lambda + |\lambda|\, K^*$ always holds (cf. (2.8)), both inclusions together with $K^* = K_\lambda^*$ show that $K = K_\lambda + |\lambda|\, K^*$ for all $-\mathrm{r} < \lambda \le 0$.

Notice that we have shown the above equality for the half-open interval $(0, -\mathrm{r}]$, and so we can apply Lemma 2.2 to get that $K = K_\lambda + |\lambda|\, K^*$ for every $\lambda \in [-\mathrm{r}, 0]$. Then Theorem 2.2 ensures that, in particular, condition (1.6) holds for $-\mathrm{r} \le \lambda \le 0$. It remains to show that K is a cap-body of $K_{-\mathrm{r}} + \mathrm{r}E$. Since $K = K_\lambda + |\lambda|\, K^*$ for all $\lambda \in [-\mathrm{r}, 0]$, by Lemma 2.5 we get

$$K_\lambda = \frac{|\lambda|}{\mathrm{r}} K_{-\mathrm{r}} + \left(1 - \frac{|\lambda|}{\mathrm{r}}\right) K.$$

By the linearity of mixed volumes (see e.g. [22, p. 279]), the above representation of K_λ can be used to compute $\mathrm{W}_{n-1}(\lambda)$ and $\mathrm{W}_{n-2}(\lambda)$ (cf. (2.1)):

$$\mathrm{W}_{n-1}(\lambda)=\mathrm{V}\bigl(K_\lambda,E[n-1]\bigr)=-\frac{\lambda}{\mathrm{r}}\mathrm{W}_{n-1}(K_{-\mathrm{r}};E)+\left(1+\frac{\lambda}{\mathrm{r}}\right)\mathrm{W}_{n-1}(K;E),$$

$$\begin{aligned}\mathrm{W}_{n-2}(\lambda)=\mathrm{V}\bigl(K_\lambda[2],E[n-2]\bigr)&=\left(\frac{\lambda}{\mathrm{r}}\right)^2\mathrm{W}_{n-2}(K_{-\mathrm{r}};E)\\&\quad-2\frac{\lambda}{\mathrm{r}}\left(1+\frac{\lambda}{\mathrm{r}}\right)\mathrm{V}\bigl(K_{-\mathrm{r}},K,E[n-2]\bigr)+\left(1+\frac{\lambda}{\mathrm{r}}\right)^2\mathrm{W}_{n-2}(K;E),\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{W}'_{n-2}(\lambda)=\frac{2}{\mathrm{r}}\biggl[\frac{\lambda}{\mathrm{r}}\mathrm{W}_{n-2}(K_{-\mathrm{r}};E)&-\left(1+2\frac{\lambda}{\mathrm{r}}\right)\mathrm{V}\bigl(K_{-\mathrm{r}},K,E[n-2]\bigr)\\&+\left(1+\frac{\lambda}{\mathrm{r}}\right)\mathrm{W}_{n-2}(K;E)\biggr].\end{aligned}$$

Since $K\in\mathcal{R}_{n-2}$ it holds $\mathrm{W}'_{n-2}(\lambda)=2\mathrm{W}_{n-1}(\lambda)$ and identifying the corresponding coefficients in the above polynomials we get, in particular,

$$\mathrm{r}\mathrm{W}_{n-1}(K;E)=\mathrm{W}_{n-2}(K;E)-\mathrm{V}\bigl(K_{-\mathrm{r}},K,E[n-2]\bigr),$$

or equivalently,

$$\mathrm{V}\bigl(K[2],E[n-2]\bigr)=\mathrm{V}\bigl(K,K_{-\mathrm{r}}+\mathrm{r}E,E[n-2]\bigr).$$

Then, using formula (2.2) for mixed volumes we get

$$\int_{\mathbb{S}^{n-1}}\bigl[h(K,u)-h(K_{-\mathrm{r}}+\mathrm{r}E,u)\bigr]\,d\mathrm{S}\bigl(K,E[n-2];u\bigr)=0,\tag{3.2}$$

and since $K_{-\mathrm{r}}+\mathrm{r}E\subseteq K$, (3.2) is equivalent to $h(K,u)=h(K_{-\mathrm{r}}+\mathrm{r}E,u)$ for all $u\in\operatorname{supp}\mathrm{S}\bigl(K,E[n-2];\cdot\bigr)=\mathrm{cl}\,\mathcal{U}_{n-2}(K)$. So K is a cap-body of $K_{-\mathrm{r}}+\mathrm{r}E$.

Now we prove the converse. Let K be a cap-body of $K_{-\mathrm{r}}+\mathrm{r}E$ satisfying (1.6) for $-\mathrm{r}\leq\lambda\leq 0$. We have to show that $K\in\mathcal{R}_{n-2}$, i.e., the real function $\mathrm{W}_i(\lambda)$ is differentiable and $\mathrm{W}'_i(\lambda)=(n-i)\mathrm{W}_{i+1}(\lambda)$ for all $i=0,\dots,n-2$ and any $-\mathrm{r}\leq\lambda\leq 0$.

By Theorem 2.2 we can write $K=K_{-\mathrm{r}}+\mathrm{r}K^*$, which implies (cf. (2.12)) that $K_\lambda=K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*$. Hence, for every $i=0,\dots,n-2$ the linearity of the mixed volumes yields

$$\begin{aligned}\mathrm{W}_i(\lambda)&=\mathrm{V}\bigl(K_\lambda[n-i],E[i]\bigr)\\&=\sum_{k=0}^{n-i}\binom{n-i}{k}(r+\lambda)^k\mathrm{V}\bigl(K_{-\mathrm{r}}[n-i-k],K^*[k],E[i]\bigr),\end{aligned}\tag{3.3}$$

which is clearly differentiable, and thus

$$\begin{aligned}\mathrm{W}'_i(\lambda)&=\sum_{k=1}^{n-i}\binom{n-i}{k}k(\mathrm{r}+\lambda)^{k-1}\mathrm{V}\bigl(K_{-\mathrm{r}}[n-i-k],K^*[k],E[i]\bigr)\\&=\sum_{k=0}^{n-i-1}\binom{n-i}{k+1}(k+1)(\mathrm{r}+\lambda)^k\mathrm{V}\bigl(K_{-\mathrm{r}}[n-i-k-1],K^*[k+1],E[i]\bigr).\end{aligned}$$

Therefore, $K\in\mathcal{R}_{n-2}$, i.e., $\mathrm{W}_i'(\lambda)=(n-i)\mathrm{W}_{i+1}(\lambda)$, if and only if (comparing coefficients of the corresponding polynomials, cf. (3.3))

$$\binom{n-i}{k+1}(k+1)\mathrm{V}\bigl(K_{-\mathrm{r}}[n-i-k-1],K^*[k+1],E[i]\bigr)\\ =(n-i)\binom{n-i-1}{k}\mathrm{V}\bigl(K_{-\mathrm{r}}[n-i-k-1],K^*[k],E[i+1]\bigr)$$

for all $i=0,\dots,n-2$ and $k=0,\dots,n-i$. Since $(k+1)\binom{n-i}{k+1}=(n-i)\binom{n-i-1}{k}$, K lies in the class $\mathcal{R}_{n-2}$ if and only if

$$\mathrm{V}\bigl(K_{-\mathrm{r}}[n-i-k-1],K^*[k],E[i+1]\bigr)=\mathrm{V}\bigl(K_{-\mathrm{r}}[n-i-k-1],K^*[k+1],E[i]\bigr) \tag{3.4}$$

for all $i=0,\dots,n-2$ and any $k=0,\dots,n-i$. Thus, in order to conclude the proof, (3.4) remains to be shown. Notice that the case $i=n-2$, $k=1$ in (3.4), i.e., the identity $\mathrm{V}\bigl(K^*,E[n-1]\bigr)=\mathrm{V}\bigl(K^*[2],E[n-2]\bigr)$, is equivalent to the fact that K^* is a cap-body of E, as we already know by Lemma 2.4.

Since $K_\lambda=K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*$ is a cap-body of $K_{-\mathrm{r}}+(\mathrm{r}+\lambda)E$ (Lemma 2.4) we can assure that

$$h(K_\lambda,u)=h\bigl(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)E,u\bigr)\quad\text{ for every }\ u\in\mathcal{U}_{n-2}(K_\lambda),$$

and so for all $u\in\operatorname{supp}\mathrm{S}\bigl(K_\lambda[n-i-1],E[i];\cdot\bigr)$, and any $i=0,\dots,n-2$. Thus, using the formula for the mixed volumes given in (2.2) we get that

$$\mathrm{V}\bigl(K_\lambda[n-i-1],K_{-\mathrm{r}}+(\mathrm{r}+\lambda)E,E[i]\bigr)=\mathrm{V}\bigl(K_\lambda[n-i],E[i]\bigr),$$

which leads to

$$\begin{aligned}\mathrm{V}\bigl(K_\lambda&[n-i-1],K_{-\mathrm{r}},E[i]\bigr)+(\mathrm{r}+\lambda)\mathrm{V}\bigl(K_\lambda[n-i-1],E[i+1]\bigr)\\&=\mathrm{V}\bigl(K_\lambda[n-i-1],K_{-\mathrm{r}}+(\mathrm{r}+\lambda)E,E[i]\bigr)=\mathrm{V}\bigl(K_\lambda[n-i],E[i]\bigr)\\&=\mathrm{V}\bigl(K_\lambda[n-i-1],K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*,E[i]\bigr)\\&=\mathrm{V}\bigl(K_\lambda[n-i-1],K_{-\mathrm{r}},E[i]\bigr)+(\mathrm{r}+\lambda)\mathrm{V}\bigl(K_\lambda[n-i-1],K^*,E[i]\bigr),\end{aligned}$$

this is,

$$\mathrm{V}\bigl(K_\lambda[n-i-1],E[i+1]\bigr)=\mathrm{V}\bigl(K_\lambda[n-i-1],K^*,E[i]\bigr).$$

Finally, writing $K_\lambda=K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*$ in the above equality and using the corresponding polynomial expressions for the mixed volumes (see [22, p. 280], cf. (1.3)), we get the identity

$$\sum_{k=0}^{n-i-1}(\mathrm{r}+\lambda)^k\mathrm{V}\bigl(K_{-\mathrm{r}}[n-i-k-1],K^*[k],E[i+1]\bigr)\\ =\sum_{k=0}^{n-i-1}(\mathrm{r}+\lambda)^k\mathrm{V}\bigl(K_{-\mathrm{r}}[n-i-k-1],K^*[k+1],E[i]\bigr)$$

for all $-\mathrm{r}\le\lambda\le 0$. Comparing coefficients we get (3.4). □

4. The main theorem and consequences

Remark 4.1. For $\rho\ge 0$ fixed, it is clear that $\mathrm{r}(K+\rho E;E)=\mathrm{r}+\rho$, where $\mathrm{r}=\mathrm{r}(K;E)$, and the inner parallel bodies of $K+\rho E$ are given by

$$(K+\rho E)_\lambda=K_{\rho+\lambda}=\begin{cases}K+(\rho+\lambda)E & \text{for } -\rho\le\lambda\le 0,\\ K\sim|\rho+\lambda|\,E & \text{for } -(\mathrm{r}+\rho)\le\lambda\le-\rho.\end{cases}$$

Then, for all $\lambda \in \big[-(\mathrm{r}+\rho),0\big]$, $\mathrm{W}_i\big((K+\rho E)_\lambda\big)$, $0\leq i\leq n$, is the i-th quermassintegral of the corresponding inner/outer parallel body of K. In particular, if $K\in\mathcal{R}_p$, $0\leq p\leq n-1$, then $\mathrm{W}_i\big((K+\rho E)_\lambda\big)$, $0\leq i\leq p$, is, up to a linear change of parameter, the i-th quermassintegral of the corresponding inner/outer parallel body of $K\in\mathcal{R}_p$; just notice that now the convex body K corresponds to $\lambda=-\rho$. This linear change of parameter ensures the needed differentiability of the quermassintegrals of $(K+\rho E)_\lambda$ for all $\lambda\in\big[-(\mathrm{r}+\rho),0\big]$. It shows that if $K\in\mathcal{R}_p$, then $K+\rho E\in\mathcal{R}_p$.

Proof of Theorem 1.1. By Remark 4.1 we may assume that (a dilation of) E is not a summand of K.

From Proposition 3.2 it follows that cap-bodies of $K_{-\mathrm{r}}+\mathrm{r}E$ satisfying condition (1.6) lie in $\mathcal{R}_{n-2}$.

Conversely, let $K\in\mathcal{R}_{n-2}\backslash\mathcal{R}_{n-1}$. Then we already know (cf. (3.1)) that

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda)=\mathrm{cl}\,\mathcal{U}_1(K_\lambda)=\cdots=\mathrm{cl}\,\mathcal{U}_{n-2}(K_\lambda),\quad \text{for all } -\mathrm{r}<\lambda\leq 0.$$

Since $K\in\mathcal{R}_{n-2}$, all inner parallel bodies $K_\lambda\in\mathcal{R}_{n-2}$, $-\mathrm{r}<\lambda\leq 0$, because their quermassintegrals satisfy the same differentiability conditions. Notice that the case $\lambda=-\mathrm{r}$ is excluded here since $K_{-\mathrm{r}}$ has no inner parallel bodies. Hence, applying Proposition 3.1 to K_λ we get

$$\mathrm{cl}\,\mathcal{U}_0(K_\lambda)=\mathcal{U}_0(K_\lambda+K_\lambda^*),\quad -\mathrm{r}<\lambda\leq 0, \tag{4.1}$$

and with Lemma 2.3 i) we conclude that

$$K\subseteq K_\lambda+|\lambda|\,K_\lambda^* \tag{4.2}$$

for every $-\mathrm{r}<\lambda\leq 0$. Since $K_\lambda+|\lambda|\,K^*\subseteq K$ always holds (cf. (2.8)), with (4.2) we obtain that for all $-\mathrm{r}<\lambda\leq 0$,

$$K_\lambda+|\lambda|\,K^*\subseteq K\subseteq K_\lambda+|\lambda|\,K_\lambda^*. \tag{4.3}$$

Notice that the left inclusion also holds for $\lambda=-\mathrm{r}$.

At this point we observe that in order to conclude the proof it suffices to show that

$$\mathrm{cl}\,\mathcal{U}_0(K)=\mathrm{cl}\,\mathcal{U}_0(K_\lambda)\quad \text{for } -\mathrm{r}<\lambda\leq 0. \tag{4.4}$$

Indeed, by (2.3) we get $\mathcal{U}_0(K^*)=\mathrm{cl}\,\mathcal{U}_0(K)=\mathrm{cl}\,\mathcal{U}_0(K_\lambda)=\mathcal{U}_0(K_\lambda^*)$, and thus $(K^*)^*=(K_\lambda^*)^*$, i.e., $K^*=K_\lambda^*$. This shows by (4.3) that $K=K_\lambda+|\lambda|\,K^*$ for $-\mathrm{r}<\lambda\leq 0$, and with Lemma 2.2 we get the validity of the identity $K=K_\lambda+|\lambda|\,K^*$ for all $-\mathrm{r}\leq\lambda\leq 0$. Then Theorem 2.2 implies that K is a tangential body of $K_{-\mathrm{r}}+\mathrm{r}E$ satisfying (1.6). Finally, Proposition 3.2 gives the required result.

So it remains to be proved (4.4) for a convex body K lying in $\mathcal{R}_{n-2}$. The inclusion $\mathrm{cl}\,\mathcal{U}_0(K_\lambda)\subseteq\mathrm{cl}\,\mathcal{U}_0(K)$ always holds for $-\mathrm{r}<\lambda\leq 0$ (cf. (2.4)) and we have to show the reverse inclusion. Thus we assume that there exists a vector $u_0\in\mathrm{cl}\,\mathcal{U}_0(K)\backslash\,\mathrm{cl}\,\mathcal{U}_0(K_{\lambda'})$ for some $\lambda'<0$. Observe that this implies that $u_0\notin\mathrm{cl}\,\mathcal{U}_0(K_\lambda)$ for all $\lambda\in[-\mathrm{r},\lambda']$. Since condition (3.1) is satisfied, such a vector $u_0\notin\mathrm{cl}\,\mathcal{U}_{n-2}(K_{\lambda'})$, i.e., u_0 is an $(n-1)$-extreme vector of $K_{\lambda'}$ which does not lie in the closure of its $(n-2)$-extreme vectors. Geometrically it corresponds to the fact that u_0 is a normal vector at a non-regular point of $K_{\lambda'}$, lying in the *interior* of the n-th dimensional normal cone of that point.

Since $u_0 \in \mathrm{cl}\,\mathcal{U}_0(K)$, there exists $\varepsilon > 0$ such that $u_0 \in \mathrm{cl}\,\mathcal{U}_0(K_\lambda)$ for all $\lambda \in (-\varepsilon, 0]$. Then, since $u_0 \notin \mathrm{cl}\,\mathcal{U}_0(K_{\lambda'})$, there exists

$$\lambda_0 = \max\{-\mathrm{r} \le \lambda < 0 : u_0 \notin \mathrm{cl}\,\mathcal{U}_0(K_\lambda)\}$$

satisfying $\lambda' \le \lambda_0 \le -\varepsilon < 0$, and so $u_0 \in \mathrm{cl}\,\mathcal{U}_0(K_\lambda)$ for all $\lambda_0 < \lambda \le 0$. Observe that if $\lambda_0 = -\mathrm{r}$ then (4.4) holds trivially, and so we may assume that $\lambda_0 > -\mathrm{r}$. Then the support function $h(K_\lambda, u_0) = h(K, u_0) + \lambda h(E, u_0)$ for all $\lambda_0 < \lambda \le 0$ (see [18, Lemma 4.4]). Again, a continuity argument (cf. proof of Lemma 2.2) leads to

$$h(K_\lambda, u_0) = h(K, u_0) + \lambda h(E, u_0) \quad \text{for all } \lambda_0 \le \lambda \le 0. \tag{4.5}$$

Since (4.1) holds, we can apply Lemma 2.3 ii) to get

$$\frac{d}{d\lambda} h(K_\lambda, u) = h(K_\lambda^*, u) \quad \text{for every } u \in \mathbb{S}^{n-1}, \tag{4.6}$$

and taking derivatives (right derivative for $\lambda = \lambda_0$) in (4.5) we get that

$$h(K_\lambda^*, u_0) = \frac{d}{d\lambda} h(K_\lambda, u_0) = h(E, u_0)$$

for all $\lambda_0 \le \lambda \le 0$. In particular, $h(K_{\lambda_0}^*, u_0) = h(E, u_0)$, which implies that u_0 cannot lie in the interior of an n-dimensional normal cone at a boundary point of K_{λ_0}, but in the boundary of the cone itself, i.e., $u_0 \in \mathrm{cl}\,\mathcal{U}_{n-2}(K_{\lambda_0}) = \mathrm{cl}\,\mathcal{U}_0(K_{\lambda_0})$, which gives the required contradiction. □

Corollary 4.1. *Every convex body lying in $\mathcal{R}_{n-2}$ has any of its inner parallel bodies (in particular its kernel) as a summand.*

Proof. Let $K \in \mathcal{R}_{n-2}$. If $K \in \mathcal{R}_{n-1}$ the conclusion is trivial (cf. (1.5)), so we assume that $K \in \mathcal{R}_{n-2} \backslash \mathcal{R}_{n-1}$.

From Theorem 1.1 it follows that K is a cap-body of $K_{-\mathrm{r}} + \mathrm{r}E$ satisfying condition (1.6), or an outer parallel body of such a set. Then by Theorem 2.2 we get that $K = K_\lambda + |\lambda|\, K^*$, $-\mathrm{r} \le \lambda \le 0$ (eventually plus ρE, for $\rho > 0$). Therefore K_λ is a summand of K for any $\lambda \in [-\mathrm{r}, 0]$. □

Notice that we can integrate relation (4.6), i.e.,

$$h(K, u) - h(K_{\lambda_0}, u) = \int_{\lambda_0}^0 h(K_\lambda^*, u)\, d\lambda = h\left(\int_{\lambda_0}^0 K_\lambda^*\, d\lambda, u\right),$$

for $-\mathrm{r} \le \lambda_0 \le 0$ and all $u \in \mathbb{S}^{n-1}$, where the last integral is taken in the Riemann-Minkowski sense (vectorial integration of functions of one real variable), for which, in this particular context, we refer to [5]. Thus we get a counterpart of Corollary 4.1, since K can be written as

$$K = K_{\lambda_0} + \int_{\lambda_0}^0 K_\lambda^*\, d\lambda, \quad \text{for all } -\mathrm{r} \le \lambda_0 \le 0.$$

This shows moreover that

$$\int_{\lambda_0}^0 K_\lambda^*\, d\lambda = |\lambda_0|\, K^*, \quad \text{for all } -\mathrm{r} \le \lambda_0 \le 0.$$

Remark 4.2. How does a convex body $K \in \mathcal{R}_{n-2}$ look like? Of course it is a cap-body of an outer parallel body of a (strictly) lower dimensional convex body. But any of these cap-bodies is not valid: the additional points which determine the set when constructing the convex hull with $K_{-\mathrm{r}} + \mathrm{r}E$ cannot lie anywhere. For instance, if $\dim K_{-\mathrm{r}} = 1$, then those points should lie in the (infinite) cylinder containing $K_{-\mathrm{r}} + \mathrm{r}E$ with $(n-2)$-dimensional spherical cross section $\mathrm{r}B_{n-2}$ (see Figure 3); otherwise the kernel $K_{-\mathrm{r}}$ would not be a summand of K and, moreover, 1-extreme normal vectors would appear when taking $K_{-\mathrm{r}}+\mathrm{r}K^*$, contradicting condition (1.6). Figure 3 shows a cap-body of $K_{-\mathrm{r}}+\mathrm{r}B_3$ lying in $\mathcal{R}_\beta$; on the contrary, the one shown in Figure 2 does not lie in $\mathcal{R}_\beta$.

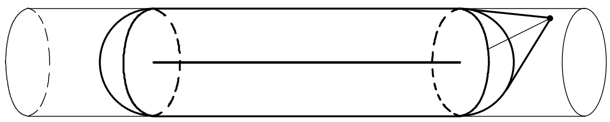

FIGURE 3. A cap-body of $K_{-\mathrm{r}} + \mathrm{r}B_3$ lying in $\mathcal{R}_\beta$.

A similar reasoning gives an idea of the situation for any dimension of the kernel. In general, if $K_{-\mathrm{r}}$ is not strictly convex in aff $K_{-\mathrm{r}}$, the "allowed" positions for the points determining the convex hull have many more restrictions, because of the segments contained in the (relative) boundary of $K_{-\mathrm{r}}$. Figure 4 shows another example of a convex body lying in $\mathcal{R}_\beta$.

5. The inradius and the roots of the Steiner polynomial

This section is devoted to prove Theorem 1.2: we will show that there are examples violating the inradius property of Conjecture 1.1 when $E = B_3$.

Proof of Theorem 1.2. Let $K \in \mathcal{K}^3$ be a convex body lying in $\mathcal{R}_\beta \backslash \mathcal{R}_\gamma$. Then it is known (see [14, Corollary 1.1]) that

$$0 \le \mathrm{V}(K_{-\lambda}) < \sum_{i=0}^{3} \binom{3}{i} \mathrm{W}_i(K; B_3)(-\lambda)^i = f_{K,B_3}(-\lambda),$$

for all $0 \le \lambda \le \mathrm{r}(K; B_3)$, the inequality being strict because $K \notin \mathcal{R}_\gamma$. Therefore, if ξ is any (positive) real root of $f_{K,B_3}(-\lambda)$ then $\xi > \mathrm{r}(K; B_3)$, which implies that $-\xi$ is a root of the Steiner polynomial $f_{K,B_3}(\lambda)$ satisfying that $-\xi < -\mathrm{r}(K; B_3)$. Thus if we find a convex body $K \in \mathcal{R}_\beta \backslash \mathcal{R}_\gamma$ such that its Steiner polynomial has 3 real roots, the above argument will show that all those (real) zeros are *strictly less* than $-\mathrm{r}(K; B_3)$, proving the theorem.

Let $C \subset \mathbb{R}^3$ be the square with edge length 1. It can be checked that the Steiner polynomial of C, $f_{C,B_3}(\lambda) = 2\lambda\bigl(1 + \pi\lambda + (2\pi/3)\lambda^2\bigr)$, has three simple real roots. In [12, Theorem 3] it is proved that the Steiner polynomial of the outer parallel body of a convex body has the same type of roots (i.e., real or complex, including multiplicities) as the Steiner polynomial of the original body. Therefore, the Steiner polynomial of the outer parallel body $C_1 = C + B_3 \in \mathcal{R}_\gamma$ has also three simple real roots. Next we consider $K = \mathrm{conv}\{C_1, p\}$, where p is a point lying on the affine hull of any diagonal of the square and close enough to C_1 (see Figure 4).

Finally observe that when the roots of a polynomial are considered as functions of the coefficients of the polynomial, these functions are continuous (see e.g. [16, p. 3]). On the other hand, in [12, Theorem 1] it is shown that the type of roots (real or complex, including multiplicities) of a Steiner polynomial is characterized by relations involving only the coefficients of the polynomial, i.e., the quermassintegrals. Moreover, quermassintegrals are also continuous functionals on $\mathcal{K}^3$ (with

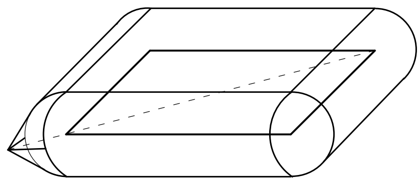

FIGURE 4. A cap-body of $C_1 = C + B_3$.

respect to the Hausdorff metric). Thus, the above three properties show that for p sufficiently close to C_1 the Steiner polynomial $f_{K,B_3}(\lambda)$ will have the same type of roots as $f_{C_1,B_3}(\lambda)$, i.e., three real roots.

It is clear that $\mathrm{r}(K; B_3) = 1$, and it is easy to verify that K is a cap-body of $K_{-1} + B_3$ satisfying (1.6) for $-1 \leq \lambda \leq 0$. Then Theorem 1.1 ensures that $K \in \mathcal{R}_\beta$, and moreover, $K \notin \mathcal{R}_\gamma$. Thus, we have constructed a convex body $K \in \mathcal{R}_\beta \backslash \mathcal{R}_\gamma$ satisfying that its Steiner polynomial has three real roots, which shows the result. □

Remark 5.1. The same argument works for an arbitrary *odd* value of the dimension, since for any convex body K lying in $\mathcal{R}_{n-2} \backslash \mathcal{R}_{n-1}$, with n odd, it holds $\mathrm{V}(K_{-\lambda}) < \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; B_n)(-\lambda)^i$ (see [14, Corollary 1.1]). The construction of the convex body $K \in \mathcal{R}_{n-2} \backslash \mathcal{R}_{n-1}$ with n real roots is analogous.

We conclude providing a numerical example in dimension $n = 3$. We use the same notation as in the proof of Theorem 1.2, and we denote by p_α the point lying on the affine hull of a diagonal of the square, so that $\alpha \in (\pi/4, \pi/2)$ is the angle determined by the diagonal containing p_α and any supporting line to C_1 through p_α. It is not difficult to check that the quermassintegrals of the body $K(\alpha) = \mathrm{conv}\{C_1, p_\alpha\}$ are given by

$$\mathrm{V}\bigl(K(\alpha)\bigr) = \frac{1}{3}\bigl(2(3 + 4\pi) + \pi g(\alpha)\bigr),$$

$$\mathrm{W}_1\bigl(K(\alpha); B_3\bigr) = \frac{1}{3}\bigl(2(1 + 3\pi) + \pi g(\alpha)\bigr),\ \mathrm{W}_2\bigl(K(\alpha); B_3\bigr) = \frac{\pi}{3}\bigl(4 + g(\alpha)\bigr),$$

where $g(\alpha) = (1 + \sin^2 \alpha)/\sin \alpha$ (see also [8, pp. 35–37]). Then if for instance $\alpha = \pi/3$, the roots of the Steiner polynomial $f_{K(\pi/3),B_3}(\lambda)$ are

$$\xi_1 = -1.011659895\ldots,\ \xi_2 = -2.099838756\ldots,\ \xi_3 = -1.404045804\ldots,$$

all of them strictly smaller than $-\mathrm{r}(K; B_3) = -1$.

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